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Traveling waves of a diffusive predator–prey model with nonlocal delay and stage structure ☆

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ABSTRACT

In this paper, a diffusive predator–prey model with nonlocal delay and stage structure is investigated. By using the cross iteration method and Schauder's fixed point theorem, we reduce the existence of traveling wave solutions to the existence of a pair of upper–lower solutions. Numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

In the natural world, many species have stage structure of immature and mature. The vital rates (rates of survival, development and reproduction and so on) are often quite different in these two stages. Recently, population models with stage structure have been taken into account (see, for example, [2,3,5,6,11,14,15]).

In [15], Xu and Ma considered the following predator–prey model with delay and stage structure for the prey

$$\begin{aligned}
\dot{x}_1(t) &= ax_2 - r_1x_1(t) - a_{11}x_1^2(t) - bx_1(t) - a_{12}x_1(t)y(t),\\
\dot{x}_2(t) &= bx_1(t) - r_2x_2(t),\\
\dot{y}(t) &= a_{21}x_1(t-\tau)y(t-\tau) - ry(t) - a_{22}y^2(t),
\end{aligned} \tag{1.1}$$

where $a > 0$ is the birth rate of immature population; $r_1 > 0$ is the death rate of the immature population; $a_{11} > 0$ is the intra-specific competition rate of the immature population; $b > 0$ is the transformation rate from the immature individuals to mature individuals; $a_{12} > 0$ is the capturing rate of the predator population; $r_2 > 0$ is the death rate of the mature population; a_2/a_{12} is the conversion rate of nutrients into the reproduction of the predator; $r > 0$ is the death rate of the predator; $a_{22} > 0$ is the intra-specific competition rate of the predator, $\tau \geqslant 0$ is a constant delay due to the gestation of the predator. In [15], the global stability of the positive equilibrium and the two boundary equilibria of the model (1.1) is discussed.

We note that system (1.1) ignored the diffusion effect. However, in many ecological systems, the species may disperse spatially as well as evolving in time. This spatial dispersal or diffusion arises from the tendency of certain species to migrate towards regions of lower population density, mainly due to resource limitation: in regions of high population density, food will become scarce, and individuals will tend to migrate to regions of lower population density.

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In realistic ecological models, delay is unavoidable. Any delays should be spatially inhomogeneous, that is, the delays affect both the temporal and spatial variables. This is due to the fact that any given individual may not necessarily have been at the same spatial location at precious times. Such delays are called *spatio-temporal delays* or *nonlocal delays*. In recent years, the effect of nonlocal delays on the dynamics of ecological models has been taken into account (see, for example, [1,7–10,12,13,16–18]).

In [9], Gourley and Kuang considered the following reaction–diffusion population model with nonlocal delay

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= d_i \frac{\partial^2 u_i}{\partial x^2} + au_m - ru_i - ae^{-r\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4d_i\pi\tau}} e^{-\frac{(x-y)^2}{4d_i\tau}} u_m(y, t-\tau)\,dy, \\
\frac{\partial u_m}{\partial t} &= d_m \frac{\partial^2 u_m}{\partial x^2} + ae^{-r\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4d_i\pi\tau}} e^{-\frac{(x-y)^2}{4d_i\tau}} u_m(y, t-\tau)\,dy - bu_m^2,
\end{aligned} \tag{1.2}$$

for $t > 0$, $x \in (-\infty, +\infty)$. The form $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4d_1\pi\tau}} e^{-\frac{(x-y)^2}{4d_1\tau}} u_m(y, t-\tau)\,dy$ represents the total number of the species u_m born at location y still alive and now at location x in the whole domain. In [9], Gourley and Kuang showed that the diffusive delay model continues to generate simple global dynamics and considered the possibility of traveling wavefront solutions of the scalar equation for the mature population, connecting the zero solution and the positive steady state of system (1.2).

Motivated by the work of Gourley and Kuang [9], Xu and Ma [15], in this paper, we are concerned with the following reaction–diffusion predator–prey model with stage structure and nonlocal delay

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + au_2(x,t) - r_1 u_1(x,t) - a_{11} u_1^2(x,t) - bu_1(x,t) - a_{12} u_1(x,t) v(x,t), \\
\frac{\partial u_2}{\partial t} &= d_1 \frac{\partial^2 u_2}{\partial x^2} + bu_1(x,t) - r_2 u_2(x,t), \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + v(x,t)\left(-r + a_{21} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4d_1\pi\tau}} e^{-\frac{(x-y)^2}{4d_1\tau}} u_1(y, t-\tau)\,dy - a_{22} v(x,t)\right),
\end{aligned} \tag{1.3}$$

for $t > 0, x \in (-\infty, +\infty)$, with initial conditions

$$\begin{aligned}
& u_1(x,\theta) = \phi(x,\theta) \geqslant 0, \\
& u_2(x,\theta) = \varphi(x,\theta) \geqslant 0, \quad (x,\theta) \in (-\infty, +\infty) \times (-\infty, 0], \\
& v(x,\theta) = \psi(x,\theta) \geqslant 0,
\end{aligned} \tag{1.4}$$

where $u_1(x,t)$, $u_2(x,t)$ and $v(x,t)$ represent the densities of immature prey, mature prey and predator populations at location x and time t, respectively. The parameters $r, r_1, r_2, a_{11}, a_{12}$, a_{21}, a_{22} are positive constants which meanings to system (1.1).

This paper is organized as follows. In Section 2, we introduce some notations and several lemmas which will be essential to our proofs. In Section 3, we discuss the existence of traveling waves of system (1.3). Finally, some numerical simulations are given to illustrate the results.

2. Preliminaries

In order to establish the existence of traveling wave solutions of system (1.3), in this section, we summarize some basic notations and concepts.

We consider the following general delayed reaction–diffusion system

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + f_1\big(u_{1t}(x), u_{2t}(x), u_{3t}(x)\big), \\
\frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + f_2\big(u_{1t}(x), u_{2t}(x), u_{3t}(x)\big), \\
\frac{\partial u_3}{\partial t} &= d_3 \frac{\partial^2 u_3}{\partial x^2} + f_3\big(u_{1t}(x), u_{2t}(x), u_{3t}(x)\big),
\end{aligned} \tag{2.1}$$

which satisfies the following hypotheses:

(A1) $f_i(0,0,0) = f_i(k_1,k_2,k_3) = 0$, $i = 1,2,3$.

(A2) There exist three positive constants $L_i > 0$ $(i = 1, 2, 3)$ such that

$$\begin{aligned}
&\left|f_1(\phi_1, \varphi_1, \psi_1) - f_1(\phi_2, \varphi_2, \psi_2)\right| \leqslant L_1\|\Phi - \Psi\|,\\
&\left|f_2(\phi_1, \varphi_1, \psi_1) - f_2(\phi_2, \varphi_2, \psi_2)\right| \leqslant L_2\|\Phi - \Psi\|,\\
&\left|f_3(\phi_1, \varphi_1, \psi_1) - f_3(\phi_2, \varphi_2, \psi_2)\right| \leqslant L_3\|\Phi - \Psi\|
\end{aligned}$$

for $\Phi = (\phi_1, \varphi_1, \psi_1)$, $\Psi = (\phi_2, \varphi_2, \psi_2) \in C([-\tau, 0], R^3)$ with $0 \leqslant \phi_1(s), \phi_2(s) \leqslant M_1$, $0 \leqslant \varphi_1(s), \varphi_2(s) \leqslant M_2$, $0 \leqslant \psi_1(s), \psi_2(s) \leqslant M_3$, $s \in [-\tau, 0]$, where $M_i > k_i$ is a positive constant, $i = 1, 2, 3$.

System (2.1) satisfies partial quasi-monotonicity conditions (PQM).
(PQM) There exist three positive constants $\beta_1, \beta_2, \beta_3 > 0$ such that

$$\begin{aligned}
&f_1(\phi_1, \varphi_1, \psi_1) - f_1(\phi_2, \varphi_2, \psi_1) + \beta_1\left[\phi_1(0) - \phi_2(0)\right] \geqslant 0,\\
&f_1(\phi_1, \varphi_1, \psi_1) - f_1(\phi_1, \varphi_1, \psi_2) \leqslant 0,\\
&f_2(\phi_1, \varphi_1, \psi_1) - f_2(\phi_2, \varphi_2, \psi_2) + \beta_2\left[\varphi_1(0) - \varphi_2(0)\right] \geqslant 0,\\
&f_3(\phi_1, \varphi_1, \psi_1) - f_1(\phi_2, \varphi_2, \psi_2) + \beta_3\left[\psi_1(0) - \psi_2(0)\right] \geqslant 0,
\end{aligned} \tag{2.2}$$

where $\phi_i, \varphi_i, \psi_i \in C([-\tau, 0], R)$, $i = 1, 2$, with $0 \leqslant \phi_2(s) \leqslant \phi_1(s) \leqslant M_1$, $0 \leqslant \varphi_2(s) \leqslant \varphi_1(s) \leqslant M_2$, $0 \leqslant \psi_2(s) \leqslant \psi_1(s) \leqslant M_3$, $s \in [-\tau, 0]$.

A traveling wave solution of (2.1) is a solution $(u_1(x,t), u_2(x,t), u_3(x,t))$ with the special form $u_1(x,t) = \phi(x+ct)$, $u_2(x,t) = \varphi(x+ct)$, $u_3(x,t) = \psi(x+ct)$ where $\phi, \varphi, \psi \in C^2(R, R^3)$ and $c > 0$ is a constant accounting for the wave speed. Substituting $u_1(x,t) = \phi(x+ct)$, $u_2(x,t) = \varphi(x+ct)$, $u_3(x,t) = \psi(x+ct)$ and denoting the traveling wave coordinate $x + ct$ still by t, we derive from (2.1) that

$$\begin{aligned}
&d_1\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t, \psi_t) = 0,\\
&d_2\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t, \psi_t) = 0,\\
&d_3\psi''(t) - c\psi'(t) + f_{c3}(\phi_t, \varphi_t, \psi_t) = 0,
\end{aligned} \tag{2.3}$$

where $f_{ci} : X_c = C([-c\tau, 0]; R^3) \to R^3$ $(i = 1, 2, 3)$ are defined by

$$\begin{aligned}
&f_{ci}(\phi, \varphi, \psi) = f_i\left(\phi^c, \varphi^c, \psi^c\right), \quad \phi^c(s) = \phi(cs),\\
&\varphi^c(s) = \varphi(cs), \qquad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0],\ i = 1, 2, 3.
\end{aligned}$$

System (2.1) has a traveling wave solution if and only if system (2.3) has a solution satisfying the following asymptotic boundary conditions,

$$\begin{aligned}
&\lim_{t\to-\infty}\phi(t) = \phi_-, \qquad \lim_{t\to-\infty}\varphi(t) = \varphi_-, \qquad \lim_{t\to-\infty}\psi(t) = \psi_-,\\
&\lim_{t\to+\infty}\phi(t) = \phi_+, \qquad \lim_{t\to+\infty}\varphi(t) = \psi_+, \qquad \lim_{t\to+\infty}\psi(t) = \psi_+.
\end{aligned} \tag{2.4}$$

Without loss of generality, we assume that $(\phi_-, \varphi_-, \psi_-) = (0, 0, 0)$ and $(\phi_+, \varphi_+, \psi_+) = (k_1, k_2, k_3)$.

We make the following hypotheses:

Definition 2.1. A pair of continuous functions $\overline{\rho} = (\overline{\phi}, \overline{\varphi}, \overline{\psi})$ and $\underline{\rho} = (\underline{\phi}, \underline{\varphi}, \underline{\psi})$ is called a pair of upper–lower solutions of system (2.3) if $\overline{\rho}$ and $\underline{\rho}$ are twice differentiable almost everywhere in R and they are essentially bounded on R, and satisfy

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + f_{c1}(\overline{\phi}_t, \overline{\varphi}_t, \underline{\psi}_t) \leqslant 0, \quad \text{a.e. in } R, \tag{2.5}$$

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}(\overline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leqslant 0, \quad \text{a.e. in } R, \tag{2.6}$$

$$d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c3}(\overline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leqslant 0, \quad \text{a.e. in } R, \tag{2.7}$$

and

$$d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + f_{c1}(\underline{\phi}_t, \underline{\varphi}_t, \overline{\psi}_t) \geqslant 0, \quad \text{a.e. in } R, \tag{2.8}$$

$$d_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_{c1}(\underline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geqslant 0, \quad \text{a.e. in } R, \tag{2.9}$$

$$d_3\underline{\psi}''(t) - c\underline{\psi}'(t) + f_{c1}(\underline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geqslant 0, \quad \text{a.e. in } R. \tag{2.10}$$

We assume that a pair of upper–lower solutions $(\overline{\phi}, \overline{\varphi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\varphi}, \underline{\psi})$ is given so that

(P1) $$(0,0,0) \leqslant (\underline{\phi}, \underline{\varphi}, \underline{\psi}) \leqslant (\overline{\phi}, \overline{\varphi}, \overline{\psi}) \leqslant (M_1, M_2, M_3), \quad t \in R.$$

(P2) $$\lim_{t\to-\infty}\big(\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t)\big) = (0,0,0),$$
$$\lim_{t\to+\infty}\big(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)\big) = \lim_{t\to+\infty}\big(\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t)\big) = (k_1, k_2, k_3).$$

Define the following profile set

$$\Gamma\big((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\overline{\phi}, \overline{\varphi}, \overline{\psi})\big) = \begin{cases} \text{(i) } \psi(t) \text{ is nondecreasing in } \mathbb{R}; \\ \text{(ii) } (\underline{\phi}, \underline{\psi})(t) \leqslant (\phi, \psi)(t) \leqslant (\overline{\phi}, \overline{\psi})(t). \end{cases}$$

It is easy to see that $\Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\overline{\phi}, \overline{\varphi}, \overline{\psi}))$ is non-empty, convex, closed and bound.

For the constants $\beta_1, \beta_2, \beta_3 > 0$ in (2.2), define $H : C(R, R^3) \to C(R, R^3)$ by

$$H_1(\phi, \varphi, \psi)(t) = f_{c1}(\phi_t, \varphi_t, \psi_t) + \beta_1 \phi(t), \quad \phi, \varphi, \psi \in C(R, R), \tag{2.11}$$
$$H_2(\phi, \varphi, \psi)(t) = f_{c2}(\phi_t, \varphi_t, \psi_t) + \beta_2 \varphi(t), \quad \phi, \varphi, \psi \in C(R, R), \tag{2.12}$$
$$H_3(\phi, \varphi, \psi)(t) = f_{c3}(\phi_t, \varphi_t, \psi_t) + \beta_3 \psi(t), \quad \phi, \varphi, \psi \in C(R, R). \tag{2.13}$$

The operators H_1, H_2 and H_3 admit the following properties:

Lemma 2.1. *Assume that* (A1) *and* (2.2) *hold, then*

$$H_1(\phi_2, \varphi_2, \psi_1)(t) \leqslant H_1(\phi_1, \varphi_1, \psi_1)(t), \qquad H_1(\phi_1, \varphi_1, \psi_1)(t) \leqslant H_1(\phi_1, \varphi_1, \psi_2)(t),$$

for $t \in R$ *with* $0 \leqslant \phi_2 \leqslant \phi_1 \leqslant M_1$, $0 \leqslant \varphi_2 \leqslant \varphi_1 \leqslant M_2$, $0 \leqslant \psi_2 \leqslant \psi_1 \leqslant M_3$.

Proof. By (2.2), a direct calculation shows that

$$H_1(\phi_1, \varphi_1, \psi_1)(t) - H_1(\phi_2, \varphi_2, \psi_1)(t) = f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{2t}, \psi_{1t}) + \beta_1\big(\phi_1(t) - \phi_2(t)\big) \geqslant 0,$$
$$H_1(\phi_1, \varphi_1, \psi_1)(t) - H_1(\phi_1, \varphi_1, \psi_2)(t) = f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{2t}) \leqslant 0. \qquad \square$$

Lemma 2.2. *Assume that* (A1) *and* (2.2) *hold, then for any* $(0,0,0) \leqslant (\phi, \varphi, \psi) \leqslant (k_1, k_2, k_3)$, *we have*

(i) $H_2(\phi, \varphi, \psi)(t) \geqslant 0$, $H_3(\phi, \varphi, \psi)(t) \geqslant 0$ *is nondecreasing for* $t \in R$.
(ii) $H_2(\phi_2, \varphi_2, \psi_2)(t) \leqslant H_2(\phi_1, \varphi_1, \psi_1)(t)$, $H_3(\phi_2, \varphi_2, \psi_2)(t) \leqslant H_3(\phi_1, \varphi_1, \psi_1)(t)$ *for* $t \in R$ *with* $0 \leqslant \phi_2 \leqslant \phi_1 \leqslant M_1$, $0 \leqslant \varphi_2 \leqslant \varphi_1 \leqslant M_2$, $0 \leqslant \psi_2 \leqslant \psi_1 \leqslant M_3$.

In terms of H_1, H_2 and H_3, system (2.3) can be rewritten as

$$\begin{aligned} &d_1\phi''(t) - c\phi'(t) - \beta_1\phi(t) + H_1(\phi, \varphi, \psi)(t) = 0, \\ &d_2\varphi''(t) - c\varphi'(t) - \beta_2\phi(t) + H_2(\phi, \varphi, \psi)(t) = 0, \\ &d_3\psi''(t) - c\psi'(t) - \beta_3\phi(t) + H_3(\phi, \varphi, \psi)(t) = 0. \end{aligned} \tag{2.14}$$

Define

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \qquad \lambda_3 = \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2},$$
$$\lambda_4 = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, \qquad \lambda_5 = \frac{c - \sqrt{c^2 + 4\beta_3 d_3}}{2d_3}, \qquad \lambda_6 = \frac{c + \sqrt{c^2 + 4\beta_3 d_3}}{2d_3}.$$

Let

$$C_K\big(R, R^3\big) = (\phi, \varphi, \psi) \in C\big(R, R^3\big): \quad (0,0,0) \leqslant (\phi, \varphi, \psi) \leqslant (M_1, M_2, M_3),$$

and define $F = (F_1, F_2, F_3) : C_K(R, R^3) \to C(R, R^3)$ by

$$F_1(\phi, \varphi, \psi)(t) = \frac{1}{d_1(\lambda_2 - \lambda_1)}\left(\int_{-\infty}^{t} e^{\lambda_1(t-s)} H_1(\phi, \varphi, \psi)(s)\,ds + \int_{t}^{\infty} e^{\lambda_2(t-s)} H_1(\phi, \varphi, \psi)(s)\,ds\right),$$

$$F_2(\phi,\varphi,\psi)(t)=\frac{1}{d_2(\lambda_4-\lambda_3)}\left(\int_{-\infty}^{t} e^{\lambda_3(t-s)}H_2(\phi,\varphi,\psi)(s)\,ds+\int_{t}^{\infty} e^{\lambda_4(t-s)}H_2(\phi,\varphi,\psi)(s)\,ds\right),$$

$$F_3(\phi,\varphi,\psi)(t)=\frac{1}{d_3(\lambda_6-\lambda_5)}\left(\int_{-\infty}^{t} e^{\lambda_5(t-s)}H_3(\phi,\varphi,\psi)(s)\,ds+\int_{t}^{\infty} e^{\lambda_6(t-s)}H_3(\phi,\varphi,\psi)(s)\,ds\right),$$

for $(\phi,\varphi,\psi)\in C_K(R,R^3)$. It is easy to see that $F_i(\phi,\varphi,\psi)$ $(i=1,2,3)$ satisfy

$$\begin{aligned}
&d_1F_1''(\phi,\varphi,\psi)-cF_1'(\phi,\varphi,\psi)-\beta_1F_1(\phi,\varphi,\psi)+H_1(\phi,\varphi,\psi)=0,\\
&d_2F_2''(\phi,\varphi,\psi)-cF_2'(\phi,\varphi,\psi)-\beta_2F_2(\phi,\varphi,\psi)+H_2(\phi,\varphi,\psi)=0,\\
&d_3F_3''(\phi,\varphi,\psi)-cF_3'(\phi,\varphi,\psi)-\beta_3F_3(\phi,\varphi,\psi)+H_3(\phi,\varphi,\psi)=0.
\end{aligned}\tag{2.15}$$

Corresponding to Lemmas 2.1 and 2.2, we have the following results.

Lemma 2.3. *Assume that* (A1) *and* (2.2) *hold, then for any* $(0,0,0)\leqslant(\phi,\varphi,\psi)\leqslant(M_1,M_2,M_3)$, *we have*

(i) $F_3(\phi,\varphi,\psi)(t)$ *is nondecreasing for* $t\in R$.

(ii)

$$\begin{aligned}
&F_1(\phi_2,\varphi_2,\psi_1)(t)\leqslant F_1(\phi_1,\varphi_1,\psi_1)(t), &&F_1(\phi_1,\varphi_1,\psi_1)(t)\leqslant F_1(\phi_1,\varphi_1,\psi_2)(t),\\
&F_2(\phi_2,\varphi_2,\psi_2)(t)\leqslant F_2(\phi_1,\varphi_1,\psi_1)(t), &&F_3(\phi_2,\varphi_2,\psi_2)(t)\leqslant F_3(\phi_1,\varphi_1,\psi_1)(t)
\end{aligned}$$

for $t\in R$ *with* $0\leqslant\phi_2\leqslant\phi_1\leqslant M_1$, $0\leqslant\varphi_2\leqslant\varphi_1\leqslant M_2$, $0\leqslant\psi_2\leqslant\psi_1\leqslant M_3$.

Lemma 2.4. *Assume that* (A2) *holds, then* $F=(F_1,F_2,F_3)$ *is continuous with respect to the norm* $|\cdot|$ *in* $B_\mu(R,R^3)$.

Lemma 2.5. *Assume that* (A1) *and* (2.2) *hold, then*

$$F\big(\Gamma\big((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\big)\big)\subset\Gamma\big((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\big).$$

Lemma 2.6. *Assume that* (2.2) *holds, then* $F:\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi}))\to\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi}))$ *is compact.*

Remark 2.1. The proofs of Lemmas 2.3–2.6 are similar to those of Lemmas 3.3–3.6 in Gan et al. [4], we omit it here.

Theorem 2.1. *Assume that* (A1), (A2) *and* (2.2) *hold. Suppose there is a pair of upper–lower solutions* $\Phi=(\overline{\phi},\overline{\varphi},\overline{\psi})$ *and* $\Psi=(\underline{\phi},\underline{\varphi},\underline{\psi})$ *for* (2.3) *satisfying* (P1) *and* (P2). *Then, system* (2.1) *has a traveling wave solution.*

Proof. Combining Lemmas 2.1–2.6 with Schauder's fixed point theorem, we know that there exists a fixed point $(\phi^*(t),\varphi^*(t),\psi^*(t))$ of F in $\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi}))$, which gives a solution of (2.3).

By (P2) and the fact that

$$(0,0,0)\leqslant(\underline{\phi},\underline{\varphi},\underline{\psi})\leqslant\big(\phi^*(t),\varphi^*(t),\psi^*(t)\big)\leqslant(\overline{\phi},\overline{\varphi},\overline{\psi})\leqslant(M_1,M_2,M_3),$$

we know that

$$\lim_{t\to-\infty}\big(\phi^*(t),\varphi^*(t),\psi^*(t)\big)=(0,0,0),\qquad \lim_{t\to+\infty}\big(\phi^*(t),\varphi^*(t),\psi^*(t)\big)=(k_1,k_2,k_3).$$

Therefore, the fixed point $(\phi^*(t),\varphi^*(t),\psi^*(t))$ satisfies the asymptotic boundary conditions (2.4). This completes the proof. □

3. Existence of traveling waves

In this section, we consider the existence of traveling wave solutions for system (1.3).

It is easy to show that system (1.3) has a steady state $E_0(0,0,0)$. If the following holds

(H1) $ab-r_2(r_1+b)>0$,

system (1.3) has a boundary steady state $E_1(u_1^*, u_2^*, 0)$, where

$$u_1^* = \frac{ab-(r_1+b)r_2}{a_{11}r_2}, \qquad u_2^* = \frac{ab^2-(r_1+b)br_2}{a_{11}r_2^2}.$$

If the following holds

(H2) $aba_{21} - a_{21}r_2(r_1+b) - a_{11}rr_2 > 0$,

system (1.3) has a positive steady state $E^*(k_1, k_2, k_3)$, where

$$\begin{aligned}
k_1 &= \frac{aba_{22}+rr_2-(r_1+b)a_{22}r_2}{a_{11}a_{22}r_2+a_{21}r_2},\\
k_2 &= \frac{ab^2a_{22}+rr_2b-(r_1+b)ba_{22}r_2}{a_{11}a_{22}r_2^2+a_{21}r_2^2},\\
k_3 &= \frac{aba_{21}-a_{21}r_2(r_1+b)-a_{11}rr_2}{a_{11}a_{22}r_2+a_{21}r_2}.
\end{aligned} \tag{3.1}$$

In this section we are interested in the possibility of a transition between the equilibria E_0 and E^* in the form of a traveling wave solution.

Lemma 3.1. *$f_{c1}(\phi_t, \varphi_t, \psi_t)$, $f_{c2}(\phi_t, \varphi_t, \psi_t)$, $f_{c3}(\phi_t, \varphi_t, \psi_t)$ of system* (1.3) *satisfy* (2.2).

Proof. For any $\phi_i, \varphi_i, \psi_i \in C([-\tau, 0], R)$ $(i = 1, 2)$, with $0 \leqslant \phi_2(s) \leqslant \phi_1(s) \leqslant M_1$, $0 \leqslant \varphi_2(s) \leqslant \varphi_1(s) \leqslant M_2$, $0 \leqslant \psi_2(s) \leqslant \psi_1(s) \leqslant M_3$, $s \in [-\tau, 0]$, we have

$$\begin{aligned}
&f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{2t}, \psi_{1t})\\
&\quad = a\varphi_1(0) - r_1\phi_1(0) - a_{11}\phi_1^2(0) - b\phi_1(0) - a_{12}\phi_1(0)\psi_1(0)\\
&\qquad - \big(a\varphi_2(0) - r_1\phi_2(0) - a_{11}\phi_2^2(0) - b\phi_2(0) - a_{12}\phi_2(0)\psi_1(0)\big)\\
&\quad = a\big(\varphi_1(0) - \varphi_2(0)\big) - r_1\big(\phi_1(0) - \phi_2(0)\big) - a_{11}\big(\phi_1(0) + \phi_2(0)\big)\big(\phi_1(0) - \phi_2(0)\big)\\
&\qquad - b\big(\phi_1(0) - \phi_2(0)\big) - a_{12}\psi_1(0)\big(\phi_1(0) - \phi_2(0)\big)\\
&\quad \geqslant (-r_1 - b - 2a_{11}M_1 - a_{12}M_3)\big(\phi_1(0) - \phi_2(0)\big).
\end{aligned}$$

Let $\beta_1 = r_1 + b + 2a_{11}M_1 + a_{12}M_3 > 0$, then it is easy to show that $f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{1t}, \psi_{2t}) + \beta_1(\phi_1(0) - \phi_2(0)) \geqslant 0$, and

$$\begin{aligned}
f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{2t}) &= a\varphi_1(0) - r_1\phi_1(0) - a_{11}\phi_1^2(0) - b\phi_1(0) - a_{12}\phi_1(0)\psi_1(0)\\
&\quad - \big(a\varphi_1(0) - r_1\phi_1(0) - a_{11}\phi_1^2(0) - b\phi_1(0) - a_{12}\phi_1(0)\psi_2(0)\big)\\
&= -a_{12}\phi_1(0)\big(\psi_1(0) - \psi_2(0)\big) \leqslant 0.
\end{aligned}$$

For $f_{c2}(\phi_t, \varphi_t, \psi_t)$, we have

$$\begin{aligned}
f_{c2}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c2}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) &= b\phi_1(0) - r_2\varphi_1(0) - \big(b\phi_2(0) - r_2\varphi_2(0)\big)\\
&\geqslant -r_2\big(\varphi_1(0) - \varphi_2(0)\big).
\end{aligned}$$

Let $\beta_2 = r_2 > 0$, then $f_{c2}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c2}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) + \beta_2(\varphi_1(0) - \varphi_2(0)) \geqslant 0$. For $f_{c3}(\phi_t, \varphi_t, \psi_t)$, we have

$$\begin{aligned}
f_{c3}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c3}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) &= \psi_1(0)\left(-r_2 + a_{21}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau}} e^{-\frac{(x-y)^2}{4d_1\tau}} \phi_1(-\tau)\, dy - a_{22}\psi_1(0)\right)\\
&\quad - \psi_2(0)\left(-r_2 + a_{21}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau}} e^{-\frac{(x-y)^2}{4d_1\tau}} \phi_2(-\tau)\, dy - a_{22}\psi_2(0)\right)\\
&\geqslant -r_2\big(\psi_1(0) - \psi_2(0)\big) - a_{22}\big(\psi_1(0) + \psi_2(0)\big)\big(\psi_1(0) - \psi_2(0)\big)\\
&\geqslant -(r_2 + 2a_{22}M_3)\big(\psi_1(0) - \psi_2(0)\big).
\end{aligned}$$

Let $\beta_3 = r_2 + 2a_{22}M_3 > 0$, then $f_{c3}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c3}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) + \beta_3(\psi_1(0) - \psi_2(0)) \geqslant 0$. This completes the proof. $\square$

In system (1.3), we choose $M_1 = \frac{aba_{22}+rr_2}{a_{21}r_2}$. Let

$$c > c^* = \max\left(2\sqrt{\frac{d_1ab}{r_2}}, 2\sqrt{d_2(-r_2+a_{21}M_1)}\right). \tag{3.2}$$

There exist $\lambda_i > 0$ $(i = 1,2,3,4)$ such that $d_1\lambda_1^2 - c\lambda_1 + \frac{ab}{r_2} = 0$, $d_2\lambda_2^2 - c\lambda_2 - r_2 + a_{21}M_1 = 0$, $d_1\lambda_3^2 - c\eta_1\lambda_3 - (r_1 + b + a_{11}M_1 + a_{12}M_3) = 0$, $d_2\lambda_4^2 - c\lambda_4 - r_2 = 0$.

We can find that there exist $\varepsilon_i > 0$ $(i = 1,2,\dots,6)$ satisfying

$$\begin{cases} a(k_2+\varepsilon_2) - (r_1+b)(k_1+\varepsilon_1) - a_{11}(k_1+\varepsilon_1)^2 < 0, \\ b\varepsilon_1 - r_2\varepsilon_2 < 0, \\ a_{21}\varepsilon_1 - a_{22}\varepsilon_3 < 0, \\ (r_1+b)\varepsilon_4 - a\varepsilon_5 + 2a_{11}\varepsilon_4 - a_{11}\varepsilon_4^2 - a_{12}k_1\varepsilon_3 + a_{12}k_3\varepsilon_4 + a_{12}\varepsilon_3\varepsilon_4 > 0, \\ -b\varepsilon_4 + r_2\varepsilon_5 > 0, \\ -\dfrac{1}{2}a_{21}(k_1+\varepsilon_4) + a_{22}\varepsilon_6 > 0. \end{cases} \tag{3.3}$$

For the above constants and suitable constants t_i $(i = 1,2,3,4) > 0$ satisfying $t_4 - c\tau > t_3 > \max(t_1,t_2)$, we define the continuous functions $\overline{\Phi}(t) = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ and $\underline{\Phi}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ as follows

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \leqslant t_1, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1, \end{cases} \qquad \overline{\varphi}(t) = \begin{cases} \frac{b}{r_2}e^{\lambda_1 t}, & t \leqslant t_1, \\ k_2 + \varepsilon_2 e^{-\lambda t}, & t > t_1, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_2 t}, & t \leqslant t_2, \\ k_3 + \varepsilon_3 e^{-\lambda t}, & t > t_2, \end{cases} \qquad \underline{\phi}(t) = \begin{cases} \eta_1 e^{\lambda_3 t}, & t \leqslant t_3, \\ k_1 - \varepsilon_4 e^{-\lambda t}, & t > t_3, \end{cases}$$

$$\underline{\varphi}(t) = \begin{cases} \frac{b}{r_2}\eta_1 e^{\lambda_3 t}, & t \leqslant t_3, \\ k_2 - \varepsilon_5 e^{-\lambda t}, & t > t_3, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} \eta_2 e^{\lambda_4 t}, & t \leqslant t_4, \\ k_3 - \varepsilon_6 e^{-\lambda t}, & t > t_4, \end{cases}$$

where $\eta_1 > 0$, $\eta_2 > 0$ are sufficiently small constants.

Lemma 3.2. *Assume that* (3.3) *holds. Then* $\Phi(t) = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ *is an upper solution of* (1.3).

Proof. For $\overline{\phi}(t)$, we need to prove that

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + a\overline{\varphi}(t) - r_1\overline{\phi}(t) - a_{11}\overline{\phi}^2(t) - b\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) \leqslant 0. \tag{3.4}$$

For $t \leqslant t_1$, in the view of $d_1\lambda_1^2 - c\lambda_1 + \frac{ab}{r_2} = 0$, we have

$$\begin{aligned} &d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + a\overline{\varphi}(t) - r_1\overline{\phi}(t) - a_{11}\overline{\phi}^2(t) - b\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) \\ &\quad \leqslant d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + a\overline{\varphi}(t) \\ &\quad = d_1e^{\lambda_1 t}\lambda_1^2 - ce^{\lambda_1 t}\lambda_1 + \frac{ab}{r_2}e^{\lambda_1 t} = 0. \end{aligned}$$

When $t \geqslant t_1$, we have

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + a\overline{\varphi}(t) - r_1\overline{\phi}(t) - a_{11}\overline{\phi}^2(t) - b\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) \leqslant I_1(\lambda),$$

where $I_1(\lambda) = d_1\varepsilon_1e^{-\lambda t}\lambda^2 + c\varepsilon_1e^{-\lambda t}\lambda + a(k_2+\varepsilon_2e^{-\lambda t}) - (r_1+b)(k_1+\varepsilon_1e^{-\lambda t}) - a_{11}(k_1+\varepsilon_1e^{-\lambda t})^2$. Then, (3.3) implies that $I_1(0) < 0$ and there exists $\lambda_2^* > 0$ such that $d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + a\overline{\varphi}(t) - r_1\overline{\phi}(t) - a_{11}\overline{\phi}^2(t) - b\overline{\phi}(t) - a_{12}\overline{\phi}(t)\underline{\psi}(t) < 0$ for all $\lambda \in (0, \lambda_1^*)$.

For $\overline{\varphi}(t)$, we need to prove that

$$d_1\overline{\varphi}''(t) - c\overline{\varphi}'(t) + b\overline{\phi}(t) - r_2\overline{\varphi}(t) \leqslant 0.$$

When $t \leqslant t_1$, we have

$$d_1\overline{\varphi}''(t) - c\overline{\varphi}'(t) + b\overline{\phi}(t) - r_2\overline{\varphi}(t) = \frac{b}{r_2}\left(d_1e^{\lambda_1 t}\lambda_1^2 - ce^{\lambda_1 t}\lambda_1\right) = -\frac{ab^2}{r_2^2}e^{\lambda_1 t} < 0.$$

When $t \geqslant t_1$, we have

$$d_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+b\overline{\phi}(t)-r_2\overline{\varphi}(t)=d_1\varepsilon_2e^{-\lambda t}\lambda^2+c\varepsilon_2e^{-\lambda t}\lambda+b\big(k_1+\varepsilon_1e^{-\lambda t}\big)-r_2\big(k_2+\varepsilon_2e^{-\lambda t}\big)=e^{-\lambda t}I_2(\lambda),$$

where $I_2(\lambda)=d_1\varepsilon_2\lambda^2+c\varepsilon_2\lambda+b\varepsilon_1-r_2\varepsilon_2$. Then, (3.3) implies that $I_2(0)<0$ and there exists $\lambda_2^*>0$ such that $d_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+b\overline{\phi}(t)-r_2\overline{\varphi}(t)<0$ for all $\lambda\in(0,\lambda_2^*)$.

For $\overline{\psi}(t)$, we need to prove that

$$d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\overline{\phi}(t-c\tau-y)\,dy-a_{22}\overline{\psi}(t)\right)\leqslant 0. \tag{3.5}$$

When $t\leqslant t_2$, we have

$$\begin{aligned}&d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\overline{\phi}(t-c\tau-y)\,dy-a_{22}\overline{\psi}(t)\right)\\&\quad\leqslant d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)(-r_2+a_{21}M_1)\\&\quad=e^{\lambda_2t}\big(d_2\lambda_2^2-c\lambda_2-r_2+a_{21}M_1\big)=0.\end{aligned}$$

When $t\geqslant t_2$, we have

$$\begin{aligned}&d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\overline{\phi}(t-c\tau-y)\,dy-a_{22}\overline{\psi}(t)\right)\\&\quad\leqslant d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)\big(-r_2+a_{21}(k_1+\varepsilon_1)-a_{22}\big(k_3+\varepsilon_3e^{-\lambda t}\big)\big)=e^{-\lambda t}I_3(\lambda),\end{aligned}$$

where $I_3(\lambda)=-d_2\varepsilon_3\lambda^2-c\varepsilon_3\lambda+(k_3+\varepsilon_3e^{-\lambda t})(a_{21}\varepsilon_1e^{\lambda t}-a_{22}\varepsilon_3)$. Then, (3.3) implies that $I_3(0)<0$ and there exists $\lambda_3^*>0$ such that $d_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\overline{\phi}(t-c\tau-y)\,dy-a_{22}\overline{\psi}(t))<0$ for all $\lambda\in(0,\lambda_3^*)$.

Let $\lambda\in(0,\min(\lambda_1^*,\lambda_2^*,\lambda_3^*))$, we see that the conclusion is true. This completes the proof. □

Lemma 3.3. *Assume that* (3.3) *holds. Then* $\Psi(t)=(\underline{\phi}(t),\underline{\varphi}(t),\underline{\psi}(t))$ *is a lower solution of* (1.3)*.*

Proof. For $\underline{\phi}(t)$, we need to prove that

$$d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+a\underline{\varphi}(t)-r_1\underline{\phi}(t)-a_{11}\underline{\phi}^2(t)-b\underline{\phi}(t)-a_{12}\underline{\phi}(t)\overline{\psi}(t)\geqslant 0. \tag{3.6}$$

When $t\leqslant t_3$, we have

$$\begin{aligned}&d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+a\underline{\varphi}(t)-r_1\underline{\phi}(t)-a_{11}\underline{\phi}^2(t)-b\underline{\phi}(t)-a_{12}\underline{\phi}(t)\overline{\psi}(t)\\&\quad\geqslant d_1\eta_1\lambda_3^2e^{\lambda_3t}-c\eta_1\lambda_3e^{\lambda_3t}-(r_1+b+a_{11}M_1+a_{12}M_3)\eta_1e^{\lambda_3t}\\&\quad=\eta_1e^{\lambda_3t}\big(d_1\lambda_3^2-c\eta_1\lambda_3-(r_1+b+a_{11}M_1+a_{12}M_3)\big)=0.\end{aligned}$$

When $t\geqslant t_3$, $\overline{\psi}(t)=k_3+\varepsilon_3e^{-\lambda t}$. We have

$$\begin{aligned}&d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+a\underline{\varphi}(t)-r_1\underline{\phi}(t)-a_{11}\underline{\phi}^2(t)-b\underline{\phi}(t)-a_{12}\underline{\phi}(t)\overline{\psi}(t)\\&\quad=-d_1\varepsilon_4\lambda^2e^{-\lambda t}-c\varepsilon_4\lambda e^{-\lambda t}+a\big(k_2-\varepsilon_5e^{-\lambda t}\big)-(r_1+b)\big(k_1-\varepsilon_4e^{-\lambda t}\big)\\&\qquad-a_{11}\big(k_1-\varepsilon_4e^{-\lambda t}\big)^2-a_{12}\big(k_1-\varepsilon_4e^{-\lambda t}\big)\big(k_3+\varepsilon_3e^{-\lambda t}\big)\\&\quad=e^{-\lambda t}I_4(\lambda),\end{aligned}$$

where $I_4(\lambda)=-d_1\varepsilon_4\lambda^2-c\varepsilon_4\lambda-a\varepsilon_5+(r_1+b)\varepsilon_4+2a_{11}k_1\varepsilon_4-a_{11}\varepsilon_4^2e^{-\lambda t}-a_{12}(k_1\varepsilon_3-k_3\varepsilon_4-\varepsilon_3\varepsilon_4e^{-\lambda t})$. Then, (3.3) implies that $I_4(0)>0$ and there exists $\lambda_4^*>0$ such that $d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+a\underline{\varphi}(t)-r_1\underline{\phi}(t)-a_{11}\underline{\phi}^2(t)-b\underline{\phi}(t)-a_{12}\underline{\phi}(t)\overline{\psi}(t)>0$ for all $\lambda\in(0,\lambda_4^*)$.

For $\underline{\varphi}(t)$, we need to prove that

$$d_1\underline{\varphi}''(t)-c\underline{\varphi}'(t)+b\underline{\phi}(t)-r_2\underline{\varphi}(t)\geqslant 0. \tag{3.7}$$

When $t\leqslant t_3$, we have

$$d_1\underline{\varphi}''(t)-c\underline{\varphi}'(t)+b\underline{\phi}(t)-r_2\underline{\varphi}(t)=d_1\eta_1\lambda_3^2e^{\lambda_3t}-c\eta_1\lambda_3e^{\lambda_3t}>0.$$

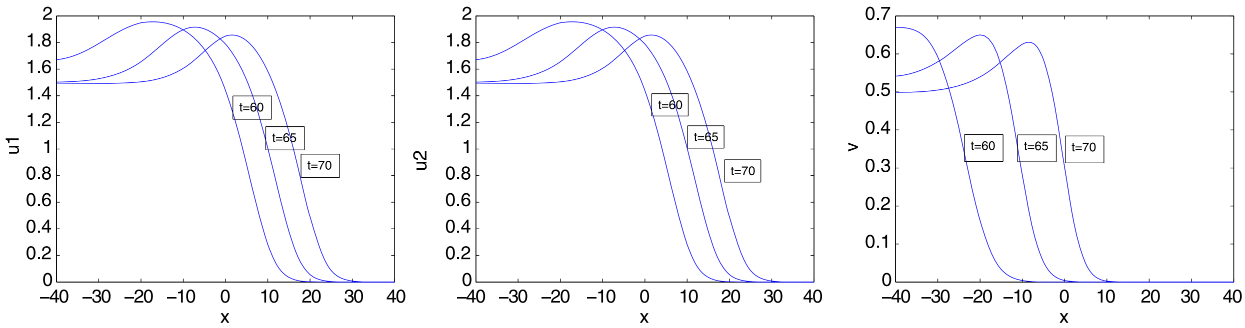

Fig. 1. The traveling wave solutions of system (1.3) with initial conditions $\phi(x,\theta)=\varphi(x,\theta)=\psi(x,\theta)=e^{-20x}/100$.

When $t \geqslant t_3$, we have

$$\begin{aligned} &d_1\underline{\varphi}''(t)-c\underline{\varphi}'(t)+b\underline{\phi}(t)-r_2\underline{\varphi}(t)\\ &\quad=-d_1\varepsilon_5\lambda^2e^{-\lambda t}-c\varepsilon_5\lambda e^{-\lambda t}+b\big(k_1-\varepsilon_4e^{-\lambda t}\big)-r_2\big(k_2-\varepsilon_5e^{-\lambda t}\big)=e^{-\lambda t}I_5(\lambda), \end{aligned}$$

where $I_5(\lambda)=-d_1\varepsilon_5\lambda^2-c\varepsilon_5\lambda-b\varepsilon_4+r_2\varepsilon_5$. Then, (3.3) implies that $I_5(0)>0$ and there exists $\lambda_5^*>0$ such that $d_1\underline{\varphi}''(t)-c\underline{\varphi}'(t)+b\underline{\phi}(t)-r_2\underline{\varphi}(t)>0$ for all $\lambda\in(0,\lambda_5^*)$.

For $\underline{\psi}(t)$, we need to prove that

$$d_2\underline{\psi}''(t)-c\underline{\psi}'(t)+\underline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\underline{\phi}(t-c\tau-y)\,dy-a_{22}\underline{\psi}(t)\right)\geqslant 0. \tag{3.8}$$

When $t \leqslant t_4$, we have

$$\begin{aligned} &d_2\underline{\psi}''(t)-c\underline{\psi}'(t)+\underline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\underline{\phi}(t-c\tau-y)\,dy-a_{22}\underline{\psi}(t)\right)\\ &\quad\geqslant d_2\eta_2\lambda_4^2e^{\lambda_4 t}-c\eta_2\lambda_4e^{\lambda_4 t}-r_2\eta_2e^{\lambda_4 t}\\ &\quad=\eta_2e^{\lambda_4 t}\big(d_2\lambda_4^2-c\lambda_4-r_2\big)=0. \end{aligned}$$

When $t \geqslant t_4$, we have

$$\begin{aligned} &d_2\underline{\psi}''(t)-c\underline{\psi}'(t)+\underline{\psi}(t)\left(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\underline{\phi}(t-c\tau-y)\,dy-a_{22}\underline{\psi}(t)\right)\\ &\quad\geqslant -d_2\varepsilon_6\lambda^2e^{-\lambda t}-c\varepsilon_6\lambda e^{-\lambda t}+\big(k_3-\varepsilon_6e^{-\lambda t}\big)\left(-r_2+\frac{1}{2}a_{21}(k_1-\varepsilon_4)-a_{22}\big(k_3-\varepsilon_6e^{-\lambda t}\big)\right)\\ &\quad=e^{-\lambda t}I_6(\lambda), \end{aligned}$$

where $I_6(\lambda)=-d_2\varepsilon_6\lambda^2-c\varepsilon_6\lambda+(k_3-\varepsilon_6e^{-\lambda t})(-\frac{1}{2}a_{21}(k_1+\varepsilon_4)\varepsilon_4e^{\lambda t}+a_{22}\varepsilon_6)$. Then, (3.3) implies that $I_6(0)>0$ and there exists $\lambda_6^*>0$ such that $d_2\underline{\psi}''(t)-c\underline{\psi}'(t)+\underline{\psi}(t)(-r_2+a_{21}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau}}e^{-\frac{y^2}{4d_1\tau}}\underline{\phi}(t-c\tau-y)\,dy-a_{22}\underline{\psi}(t))>0$ for all $\lambda\in(0,\lambda_6^*)$.

Let $\lambda\in(0,\min(\lambda_4^*,\lambda_5^*,\lambda_6^*))$, we see that the conclusion is true. This completes the proof. □

Applying Lemmas 3.1–3.3 and Theorem 2.1, we have the following conclusion.

Theorem 3.1. *Let* (H2) *hold.* c^* *is defined by* (3.2), k_1,k_2,k_3 *are defined by* (3.1)*. For every* $c>c^*$*, system* (1.3) *always has a traveling wave solution with speed c connecting the trivial steady state* $E_0(0,0,0)$ *and the positive steady state* $E^*(k_1,k_2,k_3)$*.*

4. Numerical simulations

In this section, we give a numerical simulation to illustrate the results of Section 3.

In system (1.3), we let $r=r_1=r_2=1$, $a_{11}=a_{12}=a_{21}=a_{22}=1$, $a=4$, $b=1$, $d_1=1$, $d_2=2$, $\tau=1$. Then system (1.3) has three steady states $E_0(0,0,0)$, $E_1(2,2,0)$ and $E^*(1.5,1.5,0.5)$. By Theorem 3.1, we know that the system (1.3) has a traveling wave solution which connects E_0 and E^* (see Fig. 1).

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